A Variational Approach for the Management of Green Areas

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Joint work with **D. Sciacca**

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Outline

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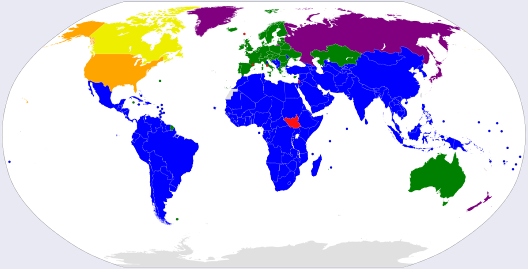
- **United States Environmental Protection Agency**: CO_2 is the primary greenhouse gas emitted through human activities
- **Intergovernmental Panel of Climate Change**: most of the observed increase in globally averaged temperatures since the mid-twentieth century is very likely due to the observed increase in anthropogenic greenhouse gas concentrations
- **Consequences**: global warming, climate change and a progressive environmental degradation (heat stress, storms and extreme precipitation, air pollution, melting glaciers, sea level rise, extinctions, reduced water resources, etc.)

Introduction

International Agreements

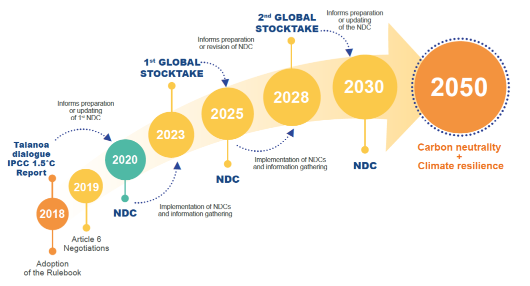
- **Kyoto Protocol (1997)**: 180 countries and more than 55% of global greenhouse gas emissions



Introduction

International Agreements

- **Kyoto Protocol (1997)**: 180 countries and more than 55% of global greenhouse gas emissions
- **Paris Agreement (2015)**: to contain the increase in the average global temperature below the 2 °C threshold (196 states)



Introduction

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- Development of cities

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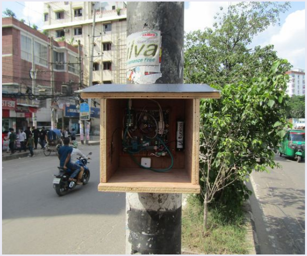
- To promote new technologies for monitoring air pollution such as air quality monitoring networks



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Good Solutions

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- To create green barriers along fast roads
- To create green roofs

Costs

- To acquire new spaces
- To train personnel
- To convert land in green areas
- To manage pre-existent types of green areas

The Mathematical Model

- n cities
- m different types of green areas
- $x_{ij} \in \mathbb{R}_+$: space of green area of type j in city i
- $\underline{x}_{ij} \in \mathbb{R}_+$: the pre-existent space of green area of type j in city i

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We assume that the optimal space of green area is not less than a quantity, l_i, imposed by law and, if in a country such a constraint does not exist, then the minimum quantity of green area imposed by law corresponds to the pre-existing one:

$$m_i := \max \left\{ l_i, \sum_{j=1}^{m} \underline{x}_{ij} \right\}, \quad \forall i = 1, \dots, n$$

The Mathematical Model

Condition 1

$$\sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} \geq m_i, \quad \forall i = 1, \ldots, n$$

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$$\sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} \geq m_i, \quad \forall i = 1, \ldots, n$$

We also impose that the total surface of green area in every city i is less than a fixed quantity u_i related to the total area of the city:

Condition 2

$$\sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} \leq u_i, \quad \forall i = 1, \ldots, n$$

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Variables and Functions

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- $c^t_{ij} = c^t_{ij}(x_{ij})$: transformation cost which is required to convert the land in green area of type j in city i

The Mathematical Model

Variables and Functions

- $f_i \in \mathbb{R}_+$: flow of urban circulation in city i
- $c_{ij}^a = c_{ij}^a(x_{ij}, f_i)$: acquisition cost associated with the expansion of green area of type j in city i
- g_i : quantity of employees to train for the management and maintenance of public green spaces
- $c_i^T = c_i^T(g_i)$: training cost for such personnel
- $c_{ij}^t = c_{ij}^t(x_{ij})$: transformation cost which is required to convert the land in green area of type j in city i
- $\gamma_{ij}^m = \gamma_{ij}^m(x_{ij}, g_i)$: management cost of green area of type j in city i

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- $c^t_{ij} = c^t_{ij}(x_{ij})$: transformation cost which is required to convert the land in green area of type j in city i
- $\gamma^m_{ij} = \gamma^m_{ij}(x_{ij}, g_i)$: management cost of green area of type j in city i
- $\underline{\gamma}^m_{ij}$: management cost of pre-existent types of green areas

The Mathematical Model

Total Management Costs

$$c_{ij}^{m} = \underline{\gamma}_{ij}^{m} + \gamma_{ij}^{m}(x_{ij}, g_i), \quad \forall i = 1, \dots, n, \ \forall j = 1, \dots, m$$

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Additional Functions

- $e_i(f_i) = \sum_{k=1}^{2} e_{ik} + e_{i3}(f_i)$: total amount of CO_2 emissions of city i

The Mathematical Model

Total Management Costs

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Additional Functions

- $e_i(f_i) = \sum_{k=1}^{2} e_{ik} + e_{i3}(f_i)$: total amount of CO_2 emissions of city i
- $e_i^a = \sum_{j=1}^{m} \alpha_j x_{ij}$: quantity of CO_2 absorbed by the overall green area in city i

The Mathematical Model

Purpose

Minimize the total costs incurred by the external institution to adapt the green area surface in each city to its real needs

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Optimality Problem

$$\min \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij}^{a}(x_{ij}, f_i) + \sum_{i=1}^{n} c_i^{T}(g_i) + \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij}^{t}(x_{ij}) + \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij}^{m}(x_{ij}, g_i) \right\}$$

The Mathematical Model

Constraints

- $x_{ij} \geq \underline{x}_{ij}, \quad \forall i, \ \forall j$
- $g_i, \ f_i \geq 0, \quad \forall i$
- $m_i \leq \sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} \leq u_i, \quad \forall i$
- $\sum_{j=1}^{m} \alpha_j x_{ij} \geq \sum_{k=1}^{2} e_{ik} + e_{i3}(f_i) - \sum_{j=1}^{m} \alpha_j \underline{x}_{ij}, \quad \forall i$
- $g_i \leq \sum_{j=1}^{m} \gamma_j x_{ij}, \quad \forall i$

Variational Inequality Formulation

Determine $(x^*, g^*, f^*) \in \mathbb{K}$ such that:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\frac{\partial c_{ij}^{a}(x_{ij}^*, f_i^*)}{\partial x_{ij}} + \frac{\partial c_{ij}^{t}(x_{ij}^*)}{\partial x_{ij}} + \frac{\partial c_{ij}^{m}(x_{ij}^*, g_i^*)}{\partial x_{ij}}\right] \times [x_{ij} - x_{ij}^*]$$

$$+ \sum_{i=1}^{n}\left[\frac{\partial c_i^{T}(g_i^*)}{\partial g_i} + \sum_{j=1}^{m}\frac{\partial c_{ij}^{m}(x_{ij}^*, g_i^*)}{\partial g_i}\right] \times [g_i - g_i^*]$$

$$+ \sum_{i=1}^{n}\left[\sum_{j=1}^{m}\frac{\partial c_{ij}^{a}(x_{ij}^*, f_i^*)}{\partial f_i}\right] \times [f_i - f_i^*] \geq 0, \quad \forall (x, g, f) \in \mathbb{K}$$

Variational Inequality Formulation

Standard Form

Find $X^* \in \mathcal{K} \subset \mathbb{R}^N$ such that $\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}$:

- $F^1_{ij}(X) \equiv \dfrac{\partial c^a_{ij}(x_{ij}, f_i)}{\partial x_{ij}} + \dfrac{\partial c^t_{ij}(x_{ij})}{\partial x_{ij}} + \dfrac{\partial c^m_{ij}(x_{ij}, g_i)}{\partial x_{ij}}$
- $F^2_i(X) \equiv \dfrac{\partial c^T_i(g_i)}{\partial g_i} + \displaystyle\sum_{j=1}^{m} \dfrac{\partial c^m_{ij}(x_{ij}, g_i)}{\partial g_i}$
- $F^3_i(X) \equiv \displaystyle\sum_{j=1}^{m} \dfrac{\partial c^a_{ij}(x_{ij}, f_i)}{\partial f_i}$

Variational Inequality Formulation

Assumptions

- Let all the involved functions be continuously differentiable and strictly convex with respect to all variables.

Variational Inequality Formulation

Assumptions

- Let all the involved functions be continuously differentiable and strictly convex with respect to all variables.
- Let F be a continuous function and the following coercivity condition be fulfilled:

$$\lim_{\substack{\|X\|\to\infty \\ x\in\mathcal{K}}} \frac{\langle F(X), X\rangle}{\|x\|} = \infty$$

Existence Results

Theorem (Existence)

Let us assume that all the Assumptions are satisfied. Then, there exists at least one solution to the variational inequality

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Theorem (Strictly Monotonicity)

The function F defining the variational inequality is strictly monotone on $\mathcal{K}$, that is:

$$\langle (F(X^1) - F(X^2))^T, X^1 - X^2 \rangle > 0, \quad \forall X^1,\ X^2 \in \mathcal{K},\ X^1 \neq X^2$$

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Theorem (Uniqueness)

As F is strictly monotone in $\mathcal{K}$, then the variational inequality admits a unique solution

Lagrange Theory

Aim

To find an alternative formulation of the variational inequality governing the minimization problem for the optimal green area model by means of the Lagrange multipliers associated with the constraints defining the feasible set $\mathcal{K}$

Lagrange Theory

We set:

- $a_i = m_i - \sum_{j=1}^{m} x_{ij} - \sum_{j=1}^{m} \underline{x}_{ij} \leq 0, \quad \forall i$
- $b_i = \sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} \underline{x}_{ij} - u_i \leq 0, \quad \forall i$
- $q_i = \sum_{k=1}^{2} e_{ik} + e_{i3}(f_i) - \sum_{j=1}^{m} \alpha_j x_{ij} - \sum_{j=1}^{m} \alpha_j \underline{x}_{ij} \leq 0, \quad \forall i$
- $k_i = g_i - \sum_{j=1}^{m} \gamma_j x_{ij} \leq 0, \quad \forall i$
- $e_{ij} = -x_{ij} + \underline{x}_{ij} \leq 0, \quad n_i = -g_i \leq 0, \quad h_i = -f_i \leq 0, \quad \forall i, \ \forall j$
- $\Gamma(X) = (a_i, b_i, q_i, k_i, e_{ij}, n_i, h_i)_{i,\ j} \quad \mathcal{K} = \{X \in \mathbb{R}_+^{2nm+2n} : \Gamma(X) \leq 0\}$

Lagrange Theory

Lagrange Function

$$\begin{aligned}\mathcal{L}(X,\omega,\varphi,\vartheta,\lambda,\psi,\mu,\varepsilon) = \quad & V(x,g,f) + \sum_{i=1}^{n} \omega_i a_i \\ & + \sum_{i=1}^{n} \varphi_i b_i + \sum_{i=1}^{n} \vartheta_i q_i + \sum_{i=1}^{n} \lambda_i k_i \\ & + \sum_{i=1}^{n} \sum_{j=1}^{m} \psi_{ij} e_{ij} + \sum_{i=1}^{n} \mu_i n_i + \sum_{i=1}^{n} \varepsilon_i h_i,\end{aligned}$$

$$\forall X \in \mathbb{R}_+^{nm+2n},\ \forall \omega \in \mathbb{R}_+^n,\ \forall \varphi \in \mathbb{R}_+^n,\ \forall \vartheta \in \mathbb{R}_+^n,$$

$$\forall \lambda \in \mathbb{R}_+^n,\ \forall \psi \in \mathbb{R}_+^{nm},\ \forall \mu \in \mathbb{R}_+^n,\ \forall \varepsilon \in \mathbb{R}_+^n$$

Lagrange Theory

Equivalence

Variational Inequality Problem $\Longleftrightarrow$ $\min_{X\in\mathcal{K}} V(x,g,f)=0$

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The minimization problem satisfies the Karush-Kuhn-Tucker conditions

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Theorem

Let X^* be the solution to the variational inequality, then the Lagrange multipliers, $\omega^* \in \mathbb{R}^n_+$, $\varphi^* \in \mathbb{R}^n_+$, $\vartheta^* \in \mathbb{R}^n_+$, $\lambda^* \in \mathbb{R}^n_+$, $\psi^* \in \mathbb{R}^{nm}_+$, $\mu^* \in \mathbb{R}^n_+$ and $\varepsilon^* \in \mathbb{R}^n_+$ associated with the constraints system do exist

Lagrange Theory

Saddle Point

$(X^*, \omega^*, \varphi^*, \vartheta^*, \lambda^*, \psi^*, \mu^*, \varepsilon^*)$ is a saddle point of the Lagrange function:

$$\begin{aligned}\mathcal{L}(X^*, \omega, \varphi, \vartheta, \lambda, \psi, \mu, \varepsilon, \lambda) \qquad & \le \mathcal{L}(X^*, \omega^*, \varphi^*, \vartheta^*, \lambda^*, \psi,^* \mu^*, \varepsilon^*) \\ & \le \mathcal{L}(X, \omega^*, \varphi^*, \vartheta^*, \lambda^*, \psi,^* \mu^*, \varepsilon^*)\end{aligned}$$

$$\forall X \in \mathbb{R}_+^{nm+2n},\ \forall \omega \in \mathbb{R}_+^n,\ \forall \varphi \in \mathbb{R}_+^n,\ \forall \vartheta \in \mathbb{R}_+^n,$$
$$\forall \lambda \in \mathbb{R}_+^n,\ \forall \psi \in \mathbb{R}_+^{nm},\ \forall \mu \in \mathbb{R}_+^n,\ \forall \varepsilon \in \mathbb{R}_+^n$$

and

$$\begin{aligned} \omega_i^* a_i^* = 0,\ \varphi_i^* b_i^* = 0, \qquad & \vartheta_i^* q_i^* = 0, \quad \forall i \\ \lambda_i^* h_i^* = 0,\ \mu_i^* n_i^* = 0, \qquad & \varepsilon_i^* h_i^* = 0, \quad \forall i \\ & \psi_{ij}^* x_{ij}^* = 0, \quad \forall i,\ \forall j \end{aligned}$$

Additional conditions

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_{ij}} &= \frac{\partial c_{ij}^{a}(x_{ij}^{*}, f_i^{*})}{\partial x_{ij}} + \frac{\partial c_{ij}^{t}(x_{ij}^{*})}{\partial x_{ij}} + \frac{\partial c_{ij}^{m}(x_{ij}^{*}, g_i^{*})}{\partial x_{ij}} \\
&\quad -\omega_i^{*} + \varphi_i^{*} - \alpha_j \vartheta_i^{*} - \gamma_j \lambda_i^{*} - \psi_{ij}^{*} = 0, \\
\frac{\partial \mathcal{L}}{\partial g_i} &= \frac{\partial c_i^{T}(g_i)}{\partial g_i} + \sum_{j=1}^{m} \frac{\partial c_{ij}^{m}(x_{ij}^{*}, g_i^{*})}{\partial g_i} + \lambda_i^{*} - \mu_i^{*} = 0, \\
\frac{\partial \mathcal{L}}{\partial f_i} &= \sum_{j=1}^{m} \frac{\partial c_{ij}^{a}(x_{ij}^{*}, f_i^{*})}{\partial f_i} + \vartheta_i^{*} \frac{\partial e_{i3}(f_i^{*})}{\partial f_i} - \varepsilon_i^{*} = 0,
\end{aligned}
$$

Lagrange Theory

Interpretation of some Lagrange Multipliers

- If $a_i^* < 0$ and $b_i^* < 0 \Longleftrightarrow m_i < \sum_{j=1}^{m} x_{ij}^* + \sum_{j=1}^{m} \underline{x}_{ij} < u_i \Longrightarrow$

  $\omega_i^* = \varphi_i^* = 0$

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- Also, if $x_{ij}^* > \underline{x}_{ij} \Longrightarrow \psi_{ij}^* = 0 \Longrightarrow$ $\frac{\partial c_{ij}^a(x_{ij}^*, f_i^*)}{\partial x_{ij}} + \frac{\partial c_{ij}^t(x_{ij}^*)}{\partial x_{ij}} + \frac{\partial c_{ij}^m(x_{ij}^*, g_i^*)}{\partial x_{ij}} = \alpha_j \vartheta_i^* + \gamma_j \lambda_i^*$

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- If $\lambda_i^* > 0$ and $\vartheta_i^* > 0 \Longrightarrow \sum_{k=1}^{2} e_{ik} + e_{i3}(f_i^*) - \sum_{j=1}^{m} \alpha_j x_{ij}^* - \sum_{j=1}^{m} \alpha_j \underline{x}_{ij} = 0$

$$g_i^* - \sum_{j=1}^{m} \gamma_j x_{ij}^* = 0$$

Lagrange Theory

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- CO_2 emissions in city i is completely absorbed by the optimal green area
- The number of employees reaches the maximum allowed number

$$\Downarrow$$

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- Each additional unit of green area or employees is unnecessary

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- The number of employees reaches the maximum allowed number

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- Each additional unit of green area or employees is unnecessary
- Similar considerations hold even if only one between λ_i^* and ϑ_i^* is positive

Lagrange Theory

Interpretation of some Lagrange Multipliers

- If both $\lambda_i^* = 0$ and $\vartheta_i^* = 0 \Longrightarrow$
$$\sum_{k=1}^{2} e_{ik} + e_{i3}(f_i^*) - \sum_{j=1}^{m} \alpha_j x_{ij}^* - \sum_{j=1}^{m} \alpha_j \underline{x}_{ij} < 0 \text{ and } g_i^* - \sum_{j=1}^{m} \gamma_j x_{ij}^* < 0$$

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- $\sum_{j=1}^{m} \left(c_{ij}^a(x_{ij}, f_i) + c_{ij}^t(x_{ij}) + c_{ij}^m(x_{ij}, g_i) \right)$ reaches its minimum value in x_{ij}^*

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- $\sum_{j=1}^{m} \left(c_{ij}^a(x_{ij}, f_i) + c_{ij}^t(x_{ij}) + c_{ij}^m(x_{ij}, g_i) \right)$ reaches its minimum value in x_{ij}^*
- If $\varphi_i^* > 0 \Longrightarrow \sum_{j=1}^{m} x_{ij}^* + \sum_{j=1}^{m} \underline{x}_{ij} = u_i$

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Interpretation of some Lagrange Multipliers

- If both $\lambda_i^* = 0$ and $\vartheta_i^* = 0 \Longrightarrow$
  $$\sum_{k=1}^{2} e_{ik} + e_{i3}(f_i^*) - \sum_{j=1}^{m} \alpha_j x_{ij}^* - \sum_{j=1}^{m} \alpha_j \underline{x}_{ij} < 0 \text{ and } g_i^* - \sum_{j=1}^{m} \gamma_j x_{ij}^* < 0$$
- $\sum_{j=1}^{m} \left(c_{ij}^a(x_{ij}, f_i) + c_{ij}^t(x_{ij}) + c_{ij}^m(x_{ij}, g_i) \right)$ reaches its minimum value in x_{ij}^*
- If $\varphi_i^* > 0 \Longrightarrow \sum_{j=1}^{m} x_{ij}^* + \sum_{j=1}^{m} \underline{x}_{ij} = u_i$
- The optimal green area added to the pre-existing one reaches the maximum percentage of city i to be allocated to green areas.

Lagrange Theory

Interpretation of some Lagrange Multipliers

- If $\psi_{ij}^* = 0 \Longrightarrow$

$$\frac{\partial c_{ij}^a(x_{ij}^*, f_i^*)}{\partial x_{ij}} + \frac{\partial c_{ij}^t(x_{ij}^*)}{\partial x_{ij}} + \frac{\partial c_{ij}^m(x_{ij}^*, g_i^*)}{\partial x_{ij}} = -\varphi_i^* + \alpha_j \vartheta_i^* + \gamma_j \lambda_i^*$$

- If $\lambda_i^* = 0$ and $\vartheta_i^* = 0 \Longrightarrow g_i^* - \sum_{j=1}^m \gamma_j x_{ij}^* < 0$

$$\sum_{k=1}^2 e_{ik} + e_{i3}(f_i^*) - \sum_{j=1}^m \alpha_j x_{ij}^* - \sum_{j=1}^m \alpha_j \underline{x}_{ij} < 0$$

- The total emissions cannot be completely absorbed by the green area of the city and therefore actions should be taken in this sense

Computational procedure

Euler Method and Convergence

$$X^{\tau+1} = P_{\mathcal{K}}(X^{\tau} - a_{\tau} F(X^{\tau})) : \sum_{\tau=0}^{\infty} a_{\tau} = \infty,\ a_{\tau} > 0,\ a_{\tau} \to 0, \text{ as } \tau \to \infty$$

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Description

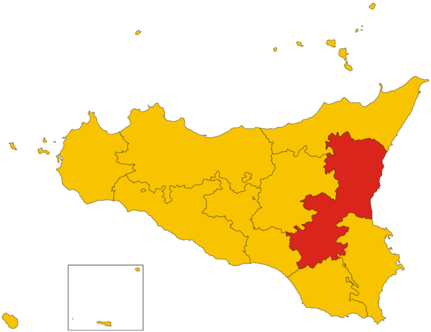
- **Step 0: Initialization** $X^0 \in \mathcal{K}$, $\tau = 1$
- **Step 1: Computation**
  $X^\tau \in \mathcal{K}$ solving the following variational inequality subproblem:
  $$\langle X^\tau + a_\tau F(X^{\tau-1}) - X^{\tau-1}, X - X^\tau \rangle \geq 0, \quad \forall X \in \mathcal{K}$$
- **Step 2: Convergence**
  $\epsilon > 0$ and check whether $|X^\tau - X^{\tau+1}| \leq \epsilon$, then stop;
  otherwise,$\tau := \tau + 1$, and go to Step 1

Numerical Examples

Type of green area	**Level of maintenance**	**Capacity of a km^2 to absorb CO_2: α_j**
$j = 1$: Urban green area	High	$\alpha_1 = 545,000$ kg/yr
$j = 2$: Natural green area	Medium	$\alpha_2 = 569,070$ kg/yr

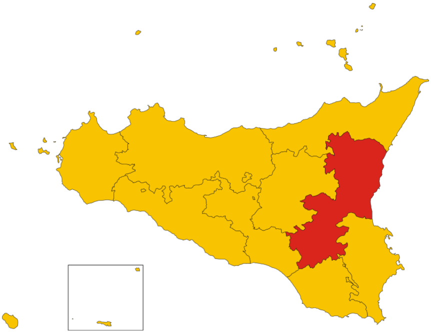
Table: Types of green areas considered in our model.

Metropolitan city of Catania



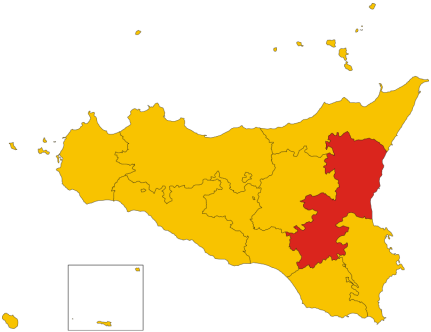
- 1,108,040 inhabitants

Metropolitan city of Catania



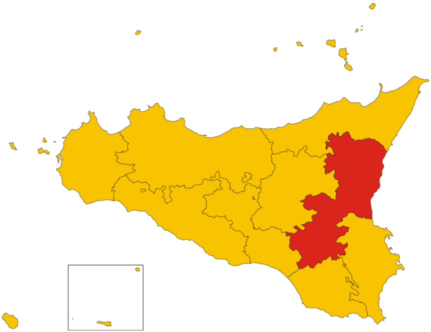
- 1,108,040 inhabitants
- 58 municipalities

Metropolitan city of Catania



- 1,108,040 inhabitants
- 58 municipalities
- 3,570 km^2 extension

Metropolitan city of Catania



- 1,108,040 inhabitants
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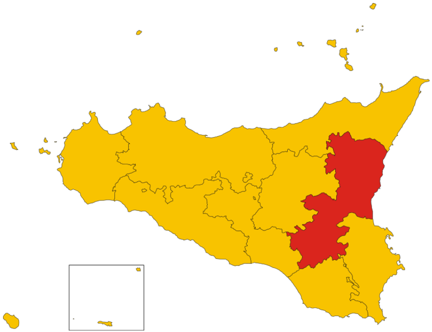
- 3,570 km^2 extension
- 9 m^2 of green area available for every citizen

Metropolitan city of Catania



- 1,108,040 inhabitants
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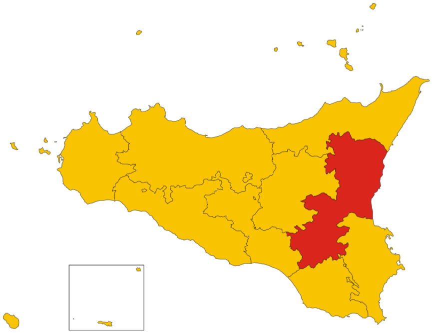
- 9.97 km^2 of green area needed

Metropolitan city of Catania



- 1,108,040 inhabitants
- 58 municipalities
- 3,570 km^2 extension
- 9 m^2 of green area available for every citizen
- 9.97 km^2 of green area needed
- 636.11 km^2 existing green area: specifically, $\underline{x}_{11} = 536.05$ km^2 and $\underline{x}_{12} = 100.06$ km^2

Metropolitan city of Catania

Emissions and Cost Functions

$$e_{11} = 637,635,890 \text{ kg/yr (emissions due to population)}$$
$$e_{12} = 728,210,000 \text{ kg/yr (emissions due to industrial activities)}$$
$$c_{11}^{a}(x_{11}, f_1) = 0.20x_{11}^2 - 0.15x_{11} + 1.20f_1^2 - 0.90f_1 + 13$$
$$c_{12}^{a}(x_{12}, f_1) = 0.30x_{12}^2 - 0.25x_{12} + 1.20f_1^2 - 0.90f_1 + 13$$
$$c_1^{T}(g_1) = 0.8g_1^2 - 0.6g_1 + 8 \;\; c_{11}^{t}(x_{11}) = 1.20x_{11}^2 - 1.10x_{11} + 8$$
$$c_{12}^{t}(x_{12}) = 1.20x_{12}^2 - 1.15x_{12} + 8$$
$$c_{11}^{m}(x_{11}, g_1) = 0.80x_{11}^2 - 0.70x_{11} + 0.8g_1^2 - 0.6g_1 + 27$$
$$c_{12}^{m}(x_{12}, g_1) = 0.50x_{12}^2 - 0.50x_{12} + 0.8g_1^2 - 0.6g_1 + 22$$

Metropolitan city of Catania

Optimal Solution

$$x_{11}^{*} = 587.27 \text{ km}^2, \quad x_{12}^{*} = 703.47 \text{ km}^2,$$
$$g_1^{*} = 140,000, \quad f_1^{*} = 710,264$$

Metropolitan city of Catania

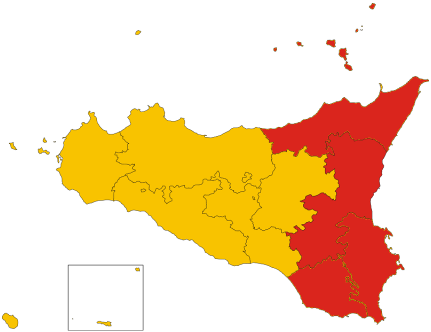
Optimal Solution

$$x_{11}^* = 587.27 \text{ km}^2, \quad x_{12}^* = 703.47 \text{ km}^2,$$
$$g_1^* = 140,000, \quad f_1^* = 710,264$$

Remark

The optimal green space amounts to 36.15% of the total territory of the metropolitan city, a value that differs greatly from the existing green area, which amounts to only 19.96% of the total territory

Eastern Sicily



- Catania
- Messina
- Syracuse
- Ragusa

Eastern Sicily

City	Surface (km^2)	Inhabithans	Emissions (kg/yr)	Pre-existing green area (km^2)
Catania	3570	1,108,040	e_{11}=637,635,890 e_{12}=728,210,000	$\underline{x}_{11}$=536 $\underline{x}_{12}$=100
Messina	3266.12	627,251	e_{21}=360,959,667 e_{22}=1,113,000,000	$\underline{x}_{21}$=48.99 $\underline{x}_{22}$=2.29
Syracuse	2124.13	397,341	e_{31}=228,654,421 e_{32}=1,020,000,000	$\underline{x}_{31}$=8.5 $\underline{x}_{32}$=120.9
Ragusa	1623.89	320,893	e_{41}=184,661,545 e_{42}=619,330,000	$\underline{x}_{41}$=6.5 $\underline{x}_{42}$=105.55

Eastern Sicily

Optimal Solution

$$x^*_{11} = 587.27 \text{ km}^2,\ x^*_{12} = 703.47 \text{ km}^2,\ g^*_1 = 140,000,\ f^*_1 = 710,264$$
$$x^*_{21} = 248.99 \text{ km}^2,\ x^*_{22} = 2.31 \text{ km}^2,\ g^*_2 = 257,600,\ f^*_2 = 341,283$$
$$x^*_{31} = 408.5 \text{ km}^2,\ x^*_{32} = 1536 \text{ km}^2,\ g^*_3 = 194,620,\ f^*_3 = 63,354$$
$$x^*_{41} = 93.5 \text{ km}^2,\ x^*_{42} = 225.98 \text{ km}^2,\ g^*_4 = 320,790,\ f^*_4 = 46,072$$

Eastern Sicily

Remark

- It is necessary to increase the percentage of green area to counteract the CO_2 emissions

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- It is necessary to increase the percentage of green area to counteract the CO_2 emissions
- Syracuse: the new green area should be 91% of the total area of the city
- The optimal flow is less than the initial flow

Conclusions

- Optimization model for the management of green areas, Lagrange theory, computational procedure, and concrete examples

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- Optimization model for the management of green areas, Lagrange theory, computational procedure, and concrete examples
- The model could be further extended and improved, by introducing also budget constraints to the local organizations or increasing the awareness of inhabitants and of the industries with respect to environment and life, requiring, for instance, that every year a part of their revenue has to be destined to the improvement and maintenance of green areas


THANK-YOU FOR YOUR ATTENTION