A variational formulation of financial networks

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Joint paper with S. Giuffré - A. Maugeri

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Outline

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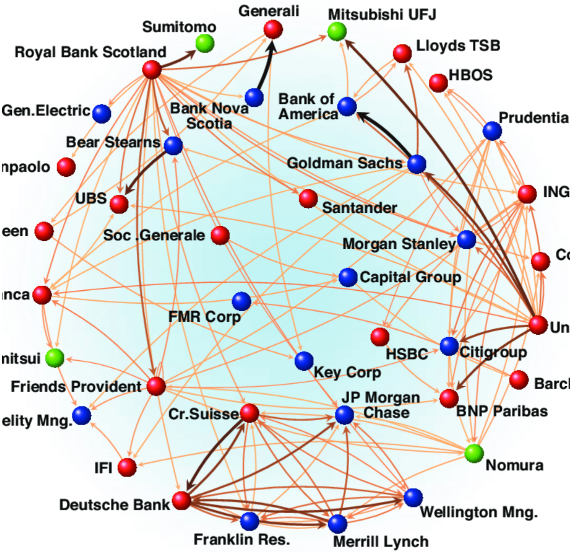
International Financial Network



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- it becomes impossible for the customer/company to obtain liquidity from other institutions.

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In general, the trend of insolvencies at global level is almost stable in 2017. The modest decline that was expected last year, equal to about a -1%, is in fact the weakest result since 2009.

State-of-the-art

The results are contained in

- G.Cappello, P.D., S. Giuffré, A. Maugeri, A variational approach to the financial problem with insolvencies and analysis of the contagion, Mathematical Analysis and Applications (T.M. Rassias, P.M. Pardalos Eds), Springer, 2019.

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The detailed financial model may be found in

- A. Barbagallo, P. D., S. Giuffré, A. Maugeri, Variational approach for a general financial equilibrium problem: The Deficit Formula, the Balance Law and the Liability Formula. A path to the economy recovery, European J. Oper. Res., 237 (1), 231-244 (2015).

State-of-the-art

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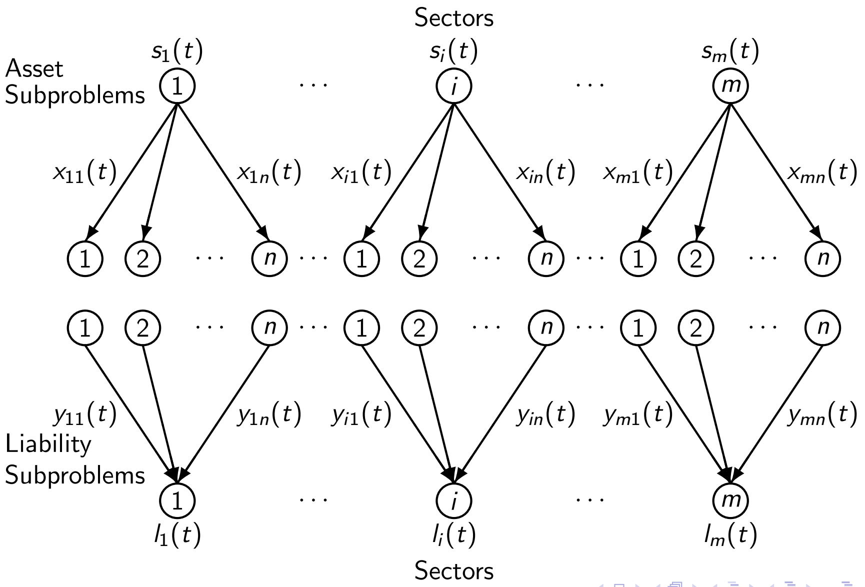
Financial Network



Functions

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Feasible sets

Feasible set for each sector i

$$P_i = \Bigg\{ (x_i(t), y_i(t)) \in L^2([0, T], \mathbb{R}^{2n}_+) : \sum_{j=1}^{n} x_{ij}(t) = s_i(t), \quad \sum_{j=1}^{n} y_{ij}(t) = l_i(t) \text{ a.e. in } [0, T] \Bigg\}$$

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Set of all feasible assets and liabilities

$$P = \left\{(x(t), y(t)) \in L^2([0,T], \mathbb{R}^{2mn}) : (x_i(t), y_i(t)) \in P_i, \ i = 1, \dots, m\right\}.$$

Feasible sets

Set of feasible instrument prices

$$\mathcal{R} = \{r \in L^2([0,T],\mathbb{R}^n) : \ \underline{r}_j(t) \le r_j(t) \le \overline{r}_j(t), \ j = 1,\dots,n, \ \text{a.e. in } [0,T]\}$$

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In this way to each investor a minimal price $\underline{r}_j$ for the assets held in the instrument j is guaranteed, whereas each investor is requested to pay for the liabilities in any case a minimal price $(1 + h_j)\underline{r}_j$.

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τ_{ij}: policy interventions in the financial equilibrium in form of taxes and price control

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Utility Function

$$U_i(t, x_i(t), y_i(t), r(t)) = \underbrace{u_i(t, x_i(t), y_i(t))}_{\text{measure of the risk of the financial agent}}$$

$$+ \underbrace{\sum_{j=1}^{n} r_j(t)(1 - \tau_{ij}(t))[x_{ij}(t) - (1 - c_j(t))(1 + h_j(t))y_{ij}(t)]}_{\text{asset holdings minus liability holdings}}$$

Assumptions (*Hp 1*)

- $U_i(t, x_i(t), y_i(t), r(t))$ *is defined on* $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, *is measurable in* t *and is continuous with respect to the other variables*

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- $\partial u_i / \partial x_{ij}$ *and* $\partial u_i / \partial y_{ij}$ *exist and they are measurable in* t *and continuous with respect to* x_i *and* y_i
- $\forall i = 1, \ldots, m$, $\forall j = 1, \ldots, n$, *and a.e. in* $[0, T]$, *the following growth conditions hold true:*

$$|u_i(t, x, y)| \leq \alpha_i(t) \|x\| \|y\|, \quad \forall x, y \in \mathbb{R}^n, \tag{1}$$

$$\left| \frac{\partial u_i(t, x, y)}{\partial x_{ij}} \right| \leq \beta_{ij}(t) \|y\|, \quad \left| \frac{\partial u_i(t, x, y)}{\partial y_{ij}} \right| \leq \gamma_{ij}(t) \|x\|, \tag{2}$$

where α_i, β_{ij}, γ_{ij} *are non-negative functions in* $L^\infty([0, T])$

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  where α_i, β_{ij}, γ_{ij} *are non-negative functions in* $L^\infty([0, T])$
- $u_i(t, x, y)$ *is concave*

Equilibrium Conditions for the price r_j

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x_{ij}^*(t)-(1-c_j(t))(1+h_j(t))y_{ij}^*(t)\right]+F_j(t)$$

$$\begin{cases} \geq 0 & \text{if } r_j^*(t)=\underline{r}_j(t) \\ =0 & \text{if } \underline{r}_j(t)<r_j^*(t)<\overline{r}_j(t) \\ \leq 0 & \text{if } r_j^*(t)=\overline{r}_j(t) \end{cases} \tag{3}$$

Equilibrium

Definition

$(x^*(t), y^*(t), r^*(t)) \in P \times \mathcal{R}$ is an equilibrium of the dynamic financial model if and only if $\forall i = 1, \dots, m$, $\forall j = 1, \dots, n$, and a.e. in $[0, T]$, it satisfies the system of inequalities and equalities

$$-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - (1 - \tau_{ij}(t)) r_j^*(t) - \mu_i^{(1)*}(t) \geq 0,$$

$$-\frac{\partial u_i(t, x^*, y^*)}{\partial y_{ij}} + (1 - \tau_{ij}(t))(1 - c_j(t))(1 + h_j(t)) r_j^*(t) - \mu_i^{(2)*}(t) \geq 0,$$

$$x_{ij}^*(t)\Big[-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - (1 - \tau_{ij}(t)) r_j^*(t) - \mu_i^{(1)*}(t)\Big] = 0,$$

$$y_{ij}^*(t)\Big[-\frac{\partial u_i(t, x^*, y^*)}{\partial y_{ij}} + (1 - \tau_{ij}(t))(1 - c_j(t))(1 + h_j(t)) r_j^*(t) - \mu_i^{(2)*}(t)\Big] = 0,$$

and verifies conditions (3) a.e. in $[0, T]$.

Variational formulation

Theorem

(x^*, y^*, r^*) is a dynamic financial equilibrium $\Leftrightarrow$ it satisfies the VI: Find $(x^*, y^*, r^*) \in P \times \mathcal{R}$:

$$\sum_{i=1}^{m} \int_0^T \Bigg\{ \sum_{j=1}^{n} \Big[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial x_{ij}} - (1-\tau_{ij}(t)) r_j^*(t) \Big] \times [x_{ij}(t) - x_{ij}^*(t)]$$
$$+ \sum_{j=1}^{n} \Big[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial y_{ij}} + (1-\tau_{ij}(t))(1-c_j(t)) r_j^*(t)(1+h_j(t)) \Big] \times [y_{ij}(t) - y_{ij}^*(t)] \Bigg\} dt$$
$$+ \sum_{j=1}^{n} \int_0^T \sum_{i=1}^{m} \Big\{ (1-\tau_{ij}(t)) \left[x_{ij}^*(t) - (1-c_j(t))(1+h_j(t)) y_{ij}^*(t) \right] + F_j(t) \Big\} \times \left[r_j(t) - r_j^*(t) \right] dt \geq 0, \qquad \forall (x, y, r) \in P \times \mathcal{R}. \tag{4}$$

Some definitions

Let X be a reflexive Banach space and let $\mathbb{K}$ be a subset of X and X^* be the dual space of X.

Definition

A mapping $A : \mathbb{K} \to X^*$ is pseudomonotone in the sense of Brezis (B-pseudomonotone) iff

1. For each sequence u_n weakly converging to u (in short $u_n \rightharpoonup u$) in $\mathbb{K}$ and such that $\limsup_n \langle Au_n, u_n - v\rangle \leq 0$ it results that:
$$\liminf_n \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle, \quad \forall v \in \mathbb{K}.$$
2. For each $v \in \mathbb{K}$ the function $u \mapsto \langle Au, u - v\rangle$ is lower bounded on the bounded subset of $\mathbb{K}$.

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Definition

A mapping $A : \mathbb{K} \to X^*$ is hemicontinuous in the sense of Fan (F-hemicontinuous) iff for all $v \in \mathbb{K}$ the function $u \mapsto \langle Au, u - v\rangle$ is weakly lower semicontinuous on $\mathbb{K}$.

Existence

Theorem

Let $\mathbb{K} \subset X$ be a nonempty closed convex bounded set and let $A : \mathbb{K} \to X^*$ be B-pseudomonotone or F-hemicontinuous. Then, variational inequality

$$\langle Au, v - u \rangle \geq 0 \quad \forall v \in \mathbb{K} \tag{5}$$

admits a solution.

Markowitz-type Function

In P.D., M. Lorino, C. Mirabella, *J. Optim. Theory Appl.*, 2016 it has been proved that Markowitz function verifies all the assumptions of the existence theorem, hence a problem with a function like this admits solutions:

$$u_i(x_i(t), y_i(t))$$

$$= \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}^T Q^i \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} + \int_0^t \begin{bmatrix} x_i(t-z) \\ y_i(t-z) \end{bmatrix}^T Q^i \begin{bmatrix} x_i(t-z) \\ y_i(t-z) \end{bmatrix} dz, \quad (6)$$

where Q^i denotes the sector i's assessment of the standard deviation of prices for each instrument j.

Infinite Dimensional Duality Theory

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- P.D., S. Giuffré, G. Idone, A. Maugeri, *Math. Ann.*, 2007
- P.D., S. Giuffré, A. Maugeri, *Communications in Applied Analysis*, 2009
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- P.D., S. Giuffré, A. Maugeri, F.Raciti, *Journal of Optimization Theory and Applications*, 2014

Lagrange Function

$$
\begin{aligned}
&\mathcal{L}(x, y, r, \lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}, \rho^{(1)}, \rho^{(2)}) = f(x, y, r) \\
&- \sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T \lambda_{ij}^{(1)}(t) x_{ij}(t)\, dt - \sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T \lambda_{ij}^{(2)} y_{ij}(t)\, dt \\
&- \sum_{i=1}^{m} \int_0^T \mu_i^{(1)}(t) \left(\sum_{j=1}^{n} x_{ij}(t) - s_i(t) \right) dt \\
&- \sum_{i=1}^{m} \int_0^T \mu_i^{(2)}(t) \left(\sum_{j=1}^{n} y_{ij}(t) - l_i(t) \right) dt \\
&+ \sum_{j=1}^{n} \int_0^T \rho_j^{(1)}(t)(\underline{r}_j(t) - r_j(t))\, dt + \sum_{j=1}^{n} \int_0^T \rho_j^{(2)}(t)(r_j(t) - \overline{r}_j(t))\, dt,
\end{aligned}
\tag{7}
$$

Lagrange Function

where

$$
\begin{aligned}
f(x,y,r) = \int_0^T \Bigg\{ & \sum_{i=1}^{m}\sum_{j=1}^{n}\left[-\frac{\partial u_i(t,x^*(t),y^*(t))}{\partial x_{ij}} - (1-\tau_{ij}(t))r_j^*(t)\right] \\
& \times[x_{ij}(t) - x_{ij}^*(t)] \\
& + \sum_{i=1}^{m}\sum_{j=1}^{n}\left[-\frac{\partial u_i(t,x^*(t),y^*(t))}{\partial y_{ij}} + (1-\tau_{ij}(t))(1-c_j(t))(1+h_j(t))r_j^*(t)\right] \\
& \times[y_{ij}(t) - y_{ij}^*(t)] \\
& + \sum_{j=1}^{n}\left[\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x_{ij}^*(t) - (1-c_j(t))(1+h_j(t))y_{ij}^*(t)\right] + F_j(t)\right] \\
& \times\left[r_j(t) - r_j^*(t)\right] \Bigg\} dt,
\end{aligned}
$$

with $(x,y,r) \in L^2([0,T],\mathbb{R}^{2mn+n})$, $\lambda^{(1)},\lambda^{(2)} \in L^2([0,T],\mathbb{R}^{mn}_+)$, $\mu^{(1)},\mu^{(2)} \in L^2([0,T],\mathbb{R}^m)$, $\rho^{(1)},\rho^{(2)} \in L^2([0,T],\mathbb{R}^n_+)$.

Theorem

Let $(x^*, y^*, r^*) \in P \times \mathcal{R}$ *be a solution to variational inequality* (4) *and let us consider the associated Lagrange functional* (7)*. Then, the strong duality holds and there exist* $\lambda^{(1)*}, \lambda^{(2)*} \in L^2([0,T], \mathbb{R}^{mn}_+)$, $\mu^{(1)*}, \mu^{(2)*} \in L^2([0,T], \mathbb{R}^m)$, $\rho^{(1)*}, \rho^{(2)*} \in L^2([0,T], \mathbb{R}^n_+)$ *such that* $(x^*, y^*, r^*, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*})$ *is a saddle point of the Lagrange functional, namely*

$$\mathcal{L}(x^*, y^*, r^*, \lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}, \rho^{(1)}, \rho^{(2)}) \tag{8}$$

$$\leq \mathcal{L}(x^*, y^*, r^*, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*}) = 0$$

$$\leq \mathcal{L}(x, y, r, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*})$$

$\forall (x, y, r) \in L^2([0,T], \mathbb{R}^{2mn+n})$, $\forall \lambda^{(1)}, \lambda^{(2)} \in L^2([0,T], \mathbb{R}^{mn}_+)$, $\forall \mu^{(1)}, \mu^{(2)} \in L^2([0,T], \mathbb{R}^m)$, $\forall \rho^{(1)}, \rho^{(2)} \in L^2([0,T], \mathbb{R}^n_+)$

and, a.e. in $[0,T]$, $\forall i=1,\dots,m$, $\forall j=1\dots,n$

$$-\frac{\partial u_i(t,x^*(t),y^*(t))}{\partial x_{ij}}-(1-\tau_{ij}(t))r_j^*(t)-\lambda_{ij}^{(1)*}(t)-\mu_i^{(1)*}(t)=0,$$

$$-\frac{\partial u_i(t,x^*(t),y^*(t))}{\partial y_{ij}}+(1-c_j(t))(1-\tau_{ij}(t))(1+h_j(t))r_j^*(t)-\lambda_{ij}^{(2)*}(t)-\mu_i^{(2)*}(t)=0,$$

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x_{ij}^*(t)-(1-c_j(t))(1+h_j(t))y_{ij}^*(t)\right]+F_j(t)+\rho_j^{(2)*}(t)=\rho_j^{(1)*}(t),$$

$$\lambda_{ij}^{(1)*}(t)x_{ij}^*(t)=0,\quad \lambda_{ij}^{(2)*}(t)y_{ij}^*(t)=0, \tag{9}$$

$$\mu_i^{(1)*}(t)\left(\sum_{j=1}^{n}x_{ij}^*(t)-s_i(t)\right)=0,\quad \mu_i^{(2)*}(t)\left(\sum_{j=1}^{n}y_{ij}^*(t)-l_i(t)\right)=0, \tag{10}$$

$$\rho_j^{(1)*}(t)(\underline{r}_j(t)-r_j^*(t))=0,\quad \rho_j^{(2)*}(t)(r_j^*(t)-\overline{r}_j(t))=0. \tag{11}$$

Deficit and Balance

Deficit Formula

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x_{ij}^{*}(t)-(1-c_j(t))(1+h_j(t))y_{ij}^{*}(t)\right]+F_j(t)+\rho_j^{(2)*}(t)=\rho_j^{(1)*}(t),$$

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Balance law

$$\sum_{i=1}^{m} l_i(t)=\sum_{i=1}^{m} s_i(t)-\sum_{i=1}^{m}\sum_{j=1}^{n}\tau_{ij}(t)\left[x_{ij}^*(t)-y_{ij}^*(t)\right]$$
$$-\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))h_j(t)y_{ij}^*(t)+\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))c_j(t)(1+h_j(t))y_{ij}^*(t)$$
$$+\sum_{j=1}^{n}F_j(t)-\sum_{j=1}^{n}\rho_j^{(1)*}(t)+\sum_{j=1}^{n}\rho_j^{(2)*}(t)$$

Deficit and Balance

Liability Formula

$$(1-c(t))\sum_{i=1}^{m} l_i(t) =$$

$$\frac{(1-\theta(t))\sum_{i=1}^{m} s_i(t) + \sum_{j=1}^{n} F_j(t) - \sum_{j=1}^{n} \rho_j^{(1)*}(t) + \sum_{j=1}^{n} \rho_j^{(2)*}(t)}{(1-\theta(t))(1+i(t))}$$

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The presence of insolvencies makes it more difficult to reach the equilibrium, because only a portion of liabilities must balance all the expenses

Duality

Dual Problem

Find $(\rho^{(1)*}, \rho^{(2)*}) \in (L^2([0, T], \mathbb{R}^n_+))^2$ such that

$$\sum_{j=1}^{n} \int_0^T (\rho_j^{(1)}(t) - \rho_j^{(1)*}(t))(\underline{r}_j(t) - r_j^*(t))dt$$
$$+ \sum_{j=1}^{n} \int_0^T (\rho_j^{(2)}(t) - \rho_j^{(2)*}(t))(r_j^*(t) - \overline{r}_j(t))dt \leq 0,$$
$$\forall (\rho^{(1)}, \rho^{(2)}) \in (L^2([0, T], \mathbb{R}^m_+))^2$$

Duality

Evaluation Index

$$E(t) = \frac{(1 - c(t)) \sum_{i=1}^{m} l_i(t)}{\sum_{i=1}^{m} \tilde{s}_i(t) + \sum_{j=1}^{n} \tilde{F}_j(t)}, \tag{12}$$

where

$$\tilde{s}_i(t) = \frac{s_i(t)}{1 + i(t)}, \quad \tilde{F}_j(t) = \frac{F_j(t)}{(1 + i(t))(1 - \theta(t))}.$$

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Remark

- *If $E(t) \geq 1$ the evaluation of the financial equilibrium is positive (better if $E(t)$ is proximal to 1);*

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Remark

- *If* $E(t) \geq 1$ *the evaluation of the financial equilibrium is positive (better if* $E(t)$ *is proximal to 1);*
- *if* $E(t) < 1$, *the evaluation of the financial equilibrium is negative.*

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- to know when it can happen
- to give a measure of it
- to understand why it occurs

Lehman Brothers' failure

Balance law

$$\sum_{i=1}^{m} l_i(t) - \sum_{i=1}^{m} s_i(t) + \sum_{i=1}^{m}\sum_{j=1}^{n} \tau_{ij}(t)\left[x_{ij}^*(t) - y_{ij}^*(t)\right]$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))h_j(t)y_{ij}^*(t) - \sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))c_j(t)(1+h_j(t))y_{ij}^*(t)$$
$$-\sum_{j=1}^{n} F_j(t) = -\sum_{j=1}^{n}\rho_j^{(1)*}(t) + \sum_{j=1}^{n}\rho_j^{(2)*}(t).$$

Balance law

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$$+\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))h_j(t)y_{ij}^*(t) - \sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))c_j(t)(1+h_j(t))y_{ij}^*(t)$$
$$-\sum_{j=1}^{n} F_j(t) = -\sum_{j=1}^{n} \rho_j^{(1)*}(t) + \sum_{j=1}^{n} \rho_j^{(2)*}(t).$$

When the left hand side is negative, it means that the sum of the liabilities, namely the investments of the system, cannot cover the expenses incurred.

Remarks

- When the right hand side is negative, it follows that the whole system is at a loss and a negative contagion is determined. Then, we can assume that the insolvencies of individual entities propagate through the entire system

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- It is sufficient that only one deficit component $\rho_j^{(1)*}(t)$ is very large to obtain, even if the other $\rho_j^{(2)*}(t)$ are lightly positive, a negative balance for the whole system
- When $\sum_{j=1}^{n} \rho_j^{(1)*}(t) > \sum_{j=1}^{n} \rho_j^{(2)*}(t) \Rightarrow E(t) \leq 1 \Rightarrow E(t)$ is a significant indicator that the financial contagion happens

Conclusions

Causes of contagion

- lack of investments

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Results

- the risk of contagion increases with the presence of insolvencies, with decreasing investments and increasing expenditure
- we suggest to reduce the insolvencies, deferring in time the payment of the liabilities, and supporting the sectors.